Note on Braverman-Gaitsgory Theory

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This note presents a detailed exposition of the Braverman-Gaitsgory theory [1]. I would like to thank my colleague, Vinit Sinha, for working through this material with me. We are especially grateful to Professor Sarah Witherspoon for her book [2], which motivated us to write this note, as well as for generously sharing her time for a helpful discussion on this topic.

1. Poincaré-Birkhoff-Witt Deformation

Let V be a k-vector space and $T(V) = \bigoplus_{i\geq 0} V^{\otimes i}$ be its tensor algebra. We note that $T(V)$ is equipped with the canonical grading

$$T(V)_i \coloneqq V^{\otimes i}$$

and the induced (increasing) filtration

$$\mathcal{F}^i T(V) \coloneqq \bigoplus_{0\leq j\leq i} V^{\otimes j}.$$

Since the filtration is induced from the grading, its associated graded algebra $\operatorname{gr} T(V)$ is canonically isomorphic to $T(V)$ itself.

Consider a subspace $P \subseteq \mathcal{F}^2 T(V)$, and let $U \coloneqq T(V)/(P)$, where (P) denotes the (two-sided) ideal generated by P. Although U is not necessarily graded, it has the induced filtration in which each $\mathcal{F}^i U$ is the image of $\mathcal{F}^i T(V)$ under the canonical projection

$$T(V) \to U.$$

By definition, each $\mathcal{F}^i T(V)$ maps onto $\mathcal{F}^i U$, and thus the induced map

$$\operatorname{gr}_i T(V) \to \operatorname{gr}_i U$$

is surjective for each i. Therefore we obtain an epimorphism

$$\operatorname{gr} T(V) \to \operatorname{gr} U.$$

Now let R be the image of P under the map

$$\mathcal{F}^2 T(V) \to \mathcal{F}^2 T(V)/\mathcal{F}^1 T(V) = \operatorname{gr}_2 T(V) \simeq T(V)_2.$$

(In other words, R is the image of P under the map

$$\mathcal{F}^2T(V) \hookrightarrow T(V) \to T(V)_2,$$

where the first map is the inclusion and the second is the projection.) Consider the following commutative diagram:

$$\begin{array}{ccccc} P & \subseteq & \mathcal{F}^2T(V) & \longrightarrow & \mathcal{F}^2U \\ & & \downarrow & & \downarrow \\ R & \subseteq & \operatorname{gr}_2 T(V) & \longrightarrow & \operatorname{gr}_2 U \end{array}$$

Since the image of P under $\mathcal{F}^2T(V) \to \mathcal{F}^2U$ (equivalently, under $T(V) \to U = T(V)/(P)$) is zero, the image of R under $T(V)_2 \simeq \operatorname{gr}_2 T(V) \to \operatorname{gr}_2 U$ is also zero. Therefore,

$$R \subseteq \operatorname{Ker}\big(T(V) \simeq \operatorname{gr} T(V) \to \operatorname{gr} U\big),$$

and we obtain the induced epimorphism

$$A \coloneqq T(V)/(R) \to \operatorname{gr} U. \tag{1}$$

If, moreover, the map (1) is injective, then $A \simeq \operatorname{gr} U$, and consequently $U \simeq \operatorname{gr} U \simeq A$ as k-vector spaces; in this case we say that U is a **Poincaré-Birkhoff-Witt (PBW) deformation** of A and that $P \subseteq \mathcal{F}^2T(V)$ is of **PBW type**.

2. Necessary Condition

Our next goal is to determine the necessary and sufficient conditions for a PBW deformation. To this end, we begin by examining the consequences of the PBW property. We start with the following general lemma, which applies to $\operatorname{gr} U$ and describes each of its graded components:

Lemma 2.1. Let T be a k-algebra with an increasing filtration and let $I \subseteq T$ be a (two-sided) ideal, so that T/I also has the induced filtration. Then

$$\operatorname{gr}_i(T/I) \simeq \mathcal{F}^iT \,/\, \left(\mathcal{F}^iT \cap (\mathcal{F}^{i-1}T + I)\right).$$

Proof. Consider the canonical projection

$$T \to T/I\,.$$

Since the inverse image of $\mathcal{F}^i(T/I)$ is $\mathcal{F}^iT + I$, we obtain

$$\begin{aligned} \operatorname{gr}_i(T/I) &= \mathcal{F}^i(T/I) \,/\, \mathcal{F}^{i-1}(T/I) \\ &\simeq (\mathcal{F}^iT + I) \,/\, (\mathcal{F}^{i-1}T + I) \\ &= \Big(\mathcal{F}^iT + (\mathcal{F}^{i-1}T + I)\Big) \,/\, (\mathcal{F}^{i-1}T + I) \\ &\simeq \mathcal{F}^iT \,/\, \left(\mathcal{F}^iT \cap (\mathcal{F}^{i-1}T + I)\right), \end{aligned}$$

which is the desired isomorphism. ∎

We will apply the lemma above for $T = T(V)$ and $I = (P)$.

Proposition 2.2. Let $P \subseteq \mathcal{F}^2T(V)$ be of PBW type. Then the following conditions hold:

(a) $P \cap \mathcal{F}^1T(V) = 0$,
(b) $(P) \cap \mathcal{F}^2T(V) = P$.

Proof. We compare A_i and $\mathrm{gr}_i U$ for $i = 0, 1, 2$.

• $i = 0$. In this case, $A_0 = \mathcal{F}^0T(V)$ and

$$\mathrm{gr}_0 U \simeq \mathcal{F}^0T(V) \,/\, \big(\mathcal{F}^0T(V) \cap (P)\big)\,.$$

Therefore

$$0 = \mathrm{Ker}(A_0 \to \mathrm{gr}_0 U) = \mathcal{F}^0T(V) \cap (P)\,. \tag{2}$$

• $i = 1$. In this case, $A_1 = \mathcal{F}^1T(V)/\mathcal{F}^0T(V)$ and

$$\mathrm{gr}_1 U \simeq \mathcal{F}^1T(V) \,/\, \big(\mathcal{F}^1T(V) \cap (\mathcal{F}^0T(V) + (P))\big)\,.$$

Therefore

$$0 = \mathrm{Ker}(A_1 \to \mathrm{gr}_1 U) = \big(\mathcal{F}^1T(V) \cap (\mathcal{F}^0T(V) + (P))\big) \,/\, \mathcal{F}^0T(V)$$

and hence

$$\mathcal{F}^1T(V) \cap (\mathcal{F}^0T(V) + (P)) = \mathcal{F}^0T(V)\,.$$

By taking the intersection with (P) on both sides, we obtain

$$\mathcal{F}^1T(V) \cap (P) = \mathcal{F}^0T(V) \cap (P) \overset{(2)}{=} 0\,, \tag{3}$$

and this proves a condition stronger than (a).

• $i = 2$. In this case,

$$A_2 = \mathcal{F}^2T(V)/(\mathcal{F}^1T(V) + R)$$

and

$$\mathrm{gr}_2 U \simeq \mathcal{F}^2T(V) \,/\, \big(\mathcal{F}^2T(V) \cap (\mathcal{F}^1T(V) + (P))\big)\,.$$

We first note that

$$\mathcal{F}^1T(V) + R = \mathcal{F}^1T(V) + P\,.$$

Therefore

$$0 = \mathrm{Ker}(A_2 \to \mathrm{gr}_2 U) = \big(\mathcal{F}^2T(V) \cap (\mathcal{F}^1T(V) + (P))\big) \,/\, (\mathcal{F}^1T(V) + P) \tag{4}$$

and hence

$$\mathcal{F}^2T(V) \cap (\mathcal{F}^1T(V) + (P)) = \mathcal{F}^1T(V) + P\,.$$

Again, taking the intersection with (P) on both sides yields

$$\mathcal{F}^2T(V) \cap (P) = (\mathcal{F}^1T(V) + P) \cap (P)\,.$$

By applying the modular law (i.e., for any submodules M, N, L of a module, $(M+N)\cap L = M\cap L + N\cap L$ if $M \subseteq L$ or $N \subseteq L$), we obtain

$$(\mathcal{F}^1T(V) + P) \cap (P) = \mathcal{F}^1T(V) \cap (P) + P \overset{(3)}{=} P\,.$$

Therefore

$$\mathcal{F}^2T(V) \cap (P) = P\,, \tag{5}$$

and this proves (b). ∎

Corollary 2.3. Let $P \subseteq \mathcal{F}^2T(V)$ be of PBW type. Then the canonical projection $P \to R$ admits an inverse of the form

$$R \to P\,, \qquad x \mapsto x - \alpha(x) - \beta(x)$$

for some linear maps $\alpha\colon R \to V$ and $\beta\colon R \to k$.

Proof. Consider the canonical projection

$$\mathcal{F}^2T(V) \to \mathcal{F}^2T(V)/\mathcal{F}^1T(V).$$

Since R is the image of P under this projection, we have

$$\operatorname{Ker}(P \to R) = P \cap \mathcal{F}^1T(V) = 0$$

by Proposition 2.2 (a). Thus the map $P \to R$ is an isomorphism and therefore has an inverse $R \to P$. Because $P \to R$ is the projection onto the homogeneous part of degree 2, this inverse map must be of the form

$$R \to P\,, \qquad x \mapsto x - \alpha(x) - \beta(x)$$

for some linear maps $\alpha\colon R \to V$ and $\beta\colon R \to k$. ∎

Corollary 2.4. Let $P \subseteq \mathcal{F}^2T(V)$ be of PBW type, and let α, β be as in Corollary 2.3. Also consider the subspace $(R \otimes_k V) \cap (V \otimes_k R) \subseteq V \otimes_k V \otimes_k V$ and the following maps

$$\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha\colon (R \otimes_k V) \cap (V \otimes_k R) \to V \otimes_k V\,,$$
$$\beta \otimes \mathrm{Id} - \mathrm{Id} \otimes \beta\colon (R \otimes_k V) \cap (V \otimes_k R) \to V\,,$$

where $k \otimes_k V$ and $V \otimes_k k$ are canonically identified with V. Then the following conditions hold:

(a) $\operatorname{Im}(\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) \subseteq R$,
(b) $\alpha \circ (\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) = -(\beta \otimes \mathrm{Id} - \mathrm{Id} \otimes \beta)$,
(c) $\beta \circ (\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) = 0$.

Proof. For any $x \in (R \otimes_k V) \cap (V \otimes_k R)$, we have

$$x - (\alpha \otimes \mathrm{Id})(x) - (\beta \otimes \mathrm{Id})(x) \in P \otimes_k V\,,$$

and analogously

$$x - (\mathrm{Id} \otimes \alpha)(x) - (\mathrm{Id} \otimes \beta)(x) \in V \otimes_k P\,.$$

Subtracting the two identities gives

$$(\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha)(x) \;+\; (\beta \otimes \mathrm{Id} - \mathrm{Id} \otimes \beta)(x) \;\in\; P \otimes_k V + V \otimes_k P \subseteq (P) \subseteq T(V)\,.$$

Since $(\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha)(x)$ and $(\beta \otimes \mathrm{Id} - \mathrm{Id} \otimes \beta)(x)$ are homogeneous of degree 2 and 1 respectively, their sum lies in $\mathcal{F}^2T(V)$ and therefore

$$(\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha)(x) \;+\; (\beta \otimes \mathrm{Id} - \mathrm{Id} \otimes \beta)(x) \in (P) \cap \mathcal{F}^2T(V) \overset{(5)}{=} P\,.$$

The three statements of the corollary now follow immediately from Corollary 2.3. ∎

Remark 2.5. To summarize, we proved that if $P \subseteq \mathcal{F}^2T(V)$ is of PBW type, then condition (a) of Proposition 2.2 together with the three conditions in Corollary 2.4 hold. The main theorem (Theorem 4.1) states that the converse is also true when $A = T(V)/(R)$ is a *Koszul algebra*, the definition of which will be given in the next section.

3. Koszul Complex

We first prove the following general statement:

Proposition 3.1. Let $R \subseteq V \otimes_k V$ be a subspace, let $A = T(V)/(R)$, and let $B_\bullet(A)$ denote the bar complex of A. Define subspaces $K'_n(A) \subseteq A^{\otimes n}$ for $n \geq 0$ by

$$K'_n(A) \coloneqq \begin{cases} k & (n=0)\,, \\ V & (n=1)\,, \\ \bigcap_{i=0}^{n-2} \left(V^{\otimes i} \otimes_k R \otimes_k V^{\otimes(n-2-i)} \right) & (n \geq 2)\,, \end{cases}$$

and set $K_n(A) \coloneqq A \otimes_k K'_n(A) \otimes_k A \subseteq B_n(A)$. Then $K_\bullet(A)$ is a subcomplex of $B_\bullet(A)$; this complex is called the **Koszul complex** of A.

Proof. We need to show that $d(K_n(A)) \subseteq K_{n-1}(A)$ for all $n \geq 1$. This is clear for $n \leq 2$, so assume that $n \geq 3$.

We first note that the image of $R \subseteq A \otimes_k A$ under the multiplication map $\mu\colon A \otimes_k A \to A$ is zero, which follows from the commutative diagram below.

$$\begin{array}{ccccc} R & \subseteq & T(V) \otimes_k T(V) & \xrightarrow{\mu_{T(V)}} & T(V) \\ & & \downarrow & & \downarrow \\ R & \subseteq & A \otimes_k A & \xrightarrow[\mu]{} & A \end{array}$$

Let $x = x_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} \in K_n(A)$. By definition of $K'_n(A)$, we have

$$x_1 \otimes \cdots \otimes x_n \in V^{\otimes i} \otimes_k R \otimes_k V^{\otimes(n-2-i)}$$

for every $0 \leq i \leq n-2$. By the remark above, the terms

$$x_0 \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n \otimes x_{n+1}$$

appearing in $d(x)$ vanish for all $1 \leq i \leq n-1$, since $x_i \otimes x_{i+1} \in R$ in each such summand. Therefore,

$$d(x) = x_0 x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes x_{n+1} + (-1)^n x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1} \otimes x_n x_{n+1}\,,$$

and both surviving terms clearly lie in $K_{n-1}(A)$. This completes the proof. ∎

In general, the Koszul complex is not necessarily exact. If $K_\bullet(A)$ is exact, then A is called a **Koszul algebra**. In this case, $K_\bullet(A)$ becomes a free resolution of A as an (A,A)-bimodule, since each $K'_n(A)$ is a k-vector space and therefore each $K_n(A)$ is a free (A,A)-bimodule.

3.1 Quasi-isomorphisms between $K_\bullet(A)$ and $\overline{B}_\bullet(A)$

Let $A = T(V)/(R)$ be a Koszul algebra. Then the inclusion $\iota_\bullet \colon K_\bullet(A) \hookrightarrow B_\bullet(A)$ is a chain map between resolutions lifting the identity map on A, hence it is a quasi-isomorphism.

$$\begin{array}{ccc} K_\bullet(A) & \xhookrightarrow{\iota_\bullet} & B_\bullet(A) \\ \downarrow & & \downarrow \\ A & = & A \end{array}$$

Moreover, since A is a graded algebra, each $K_n(A)$ and $B_n(A)$ is graded by the grading induced from A, and each $\iota_n \colon K_n(A) \to B_n(A)$ is degree preserving.

We further compose $\iota_\bullet$ with the canonical projection $q_\bullet \colon B_\bullet(A) \to \overline{B}_\bullet(A)$ from the bar complex to the reduced bar complex. Recall that

$$\operatorname{Ker}(q_n) = \sum_{i=1}^{n} A^{\otimes i} \otimes_k k \otimes_k A^{\otimes n-i+1} \subseteq A^{\otimes (n+2)} \,.$$

Since $K_n(A) \cap \operatorname{Ker}(q_n) = 0$ for each n, the composition $q_\bullet \circ \iota_\bullet \colon K_\bullet(A) \to \overline{B}_\bullet(A)$ is a monomorphism, and the Koszul complex maps isomorphically onto its image.

Lemma 3.2. There exists a chain map $\overline{\psi}_\bullet \colon \overline{B}_\bullet(A) \to K_\bullet(A)$ lifting the identity map on A (and therefore a quasi-isomorphism) such that (i) $\overline{\psi}_\bullet \circ (q_\bullet \iota_\bullet) = \mathrm{Id}_{K_\bullet(A)}$, and (ii) each $\overline{\psi}_n \colon \overline{B}_n(A) \to K_n(A)$ is degree preserving.

$$\begin{array}{ccc} K_\bullet(A) & \xrightarrow{\iota_\bullet} & B_\bullet(A) \\ & \overset{\overline{\psi}_\bullet}{\nwarrow} & \downarrow{\scriptstyle q_\bullet} \\ & & \overline{B}_\bullet(A) \end{array}$$

The lemma above is a consequence of the following general result:

Proposition 3.3. Let M be an object of an abelian category and $P_\bullet$, $Q_\bullet$ be projective resolutions of M such that $Q_\bullet$ is a subcomplex of $P_\bullet$ and each Q_n is a direct summand of P_n. If we denote the monomorphism by $\iota_\bullet \colon Q_\bullet \to P_\bullet$, then there exists a chain map $\psi_\bullet \colon P_\bullet \to Q_\bullet$ lifting the identity on M satisfying $\psi_\bullet \circ \iota_\bullet = \mathrm{Id}_{Q_\bullet}$.

Proof. Since each Q_n is a direct summand of P_n, we can choose a subobject L_n of P_n such that $P_n = Q_n \oplus L_n$. (Note that the subobjects $L_\bullet$ do NOT form a subcomplex of $P_\bullet$, since $d(L_n)$ is not necessarily contained in L_{n-1}.)

We construct morphisms $\psi'_n \colon L_n \to Q_n$ inductively so that

$$\psi_n \colon P_n = Q_n \oplus L_n \to Q_n$$

induced from $\mathrm{Id}_{Q_n} \colon Q_n \to Q_n$ and ψ'_n assemble into a chain map $\psi_\bullet \colon P_\bullet \to Q_\bullet$ lifting the identity on M.

• $n = 0$.

$$\begin{array}{ccccc} L_0 & \longrightarrow & M & \longrightarrow & 0 \\ {\scriptstyle \psi'_0}\downarrow & & \| & & \\ Q_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Since L_0 is projective, the map $L_0 \to M$ admits a lift $\psi'_0 \colon L_0 \to Q_0$.

• $n > 0$.

$$\begin{array}{ccccc} L_n & \xrightarrow{d} & P_{n-1} & \xrightarrow{d} & P_{n-2} \\ {\scriptstyle \psi'_n}\downarrow & & \downarrow{\scriptstyle \psi_{n-1}} & & \downarrow{\scriptstyle \psi_{n-2}} \\ Q_n & \xrightarrow[d]{} & Q_{n-1} & \xrightarrow[d]{} & Q_{n-2} \end{array}$$

Since

$$d\,\psi_{n-1}\,d = \psi_{n-2}\,d^2 = 0 \colon L_n \to Q_{n-2}\,,$$

it factors through

$$\operatorname{Ker}(d \colon Q_{n-1} \to Q_{n-2}) = \operatorname{Im}(d \colon Q_n \to Q_{n-1})\,.$$

Therefore, by projectivity of L_n, there exists a lift

$$\psi'_n \colon L_n \to Q_n$$

of $\psi_{n-1}d$.

Having constructed all ψ'_n, the maps ψ_n assemble into a chain map $\psi_\bullet \colon P_\bullet \to Q_\bullet$ lifting the identity on M. Since $\psi_\bullet \circ \iota_\bullet = \mathrm{Id}_{Q_\bullet}$ by construction, this completes the proof. ■

Proof of Lemma 3.2. Let $\mathcal{C}$ be the abelian category of all (A, A)-bimodules, and let $\mathcal{C}_{\mathrm{gr}}$ be the abelian category consisting of all graded (A, A)-bimodules as objects and degree preserving maps as morphisms. We first show that any object P in $\mathcal{C}_{\mathrm{gr}}$ which is projective in $\mathcal{C}$ is also projective in $\mathcal{C}_{\mathrm{gr}}$.

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle \gamma}\swarrow & \downarrow{\scriptstyle \beta} & & \\ M & \xrightarrow[\alpha]{} & N & \longrightarrow & 0 \end{array}$$

Let $M, N \in \mathcal{C}_{\mathrm{gr}}$ and let $\alpha \colon M \to N$ be an epimorphism in $\mathcal{C}_{\mathrm{gr}}$ (hence surjective). Since α is also an epimorphism in $\mathcal{C}$, the map $\beta \in \operatorname{Hom}_{\mathcal{C}_{\mathrm{gr}}}(P, N)$ has a lift $\gamma \in \operatorname{Hom}_{\mathcal{C}}(P, M)$. Let γ_0 denote the degree zero component of γ. Since α, β are degree preserving, taking the degree zero component of the equality $\beta = \alpha \circ \gamma$ yields $\beta = \alpha \circ \gamma_0$, and therefore P is also projective in $\mathcal{C}_{\mathrm{gr}}$.

The previous argument shows that $\overline{B}_\bullet(A)$ and $K_\bullet(A)$ are projective resolutions in $\mathcal{C}_{\mathrm{gr}}$ as well, and now Lemma 3.2 follows directly from Proposition 3.3 applied to $\mathcal{C}_{\mathrm{gr}}$. ■

4. Poincaré-Birkhoff-Witt Theorem

We now state and prove the main theorem:

Theorem 4.1. Let V be a k-vector space, $R \subseteq V \otimes_k V$ be a subspace, and $A = T(V)/(R)$ be the graded algebra equipped with the canonical grading induced from $T(V)$. Assume that A is a Koszul algebra.

Let $\alpha\colon R \to V$ and $\beta\colon R \to k$ be k-linear maps,

$$P \coloneqq \{x - \alpha(x) - \beta(x) \in T(V) \,|\, x \in R\}$$

be a subspace of $\mathcal{F}^2 T(V)$, and $U = T(V)/(P)$ be the filtered algebra equipped with the canonical filtration induced from $T(V)$.

Also consider the subspace $(R \otimes_k V) \cap (V \otimes_k R) \subseteq V \otimes_k V \otimes_k V$ and the maps

$$\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha\colon (R \otimes_k V) \cap (V \otimes_k R) \to V \otimes_k V\,,$$
$$\beta \otimes \mathrm{Id} - \mathrm{Id} \otimes \beta\colon (R \otimes_k V) \cap (V \otimes_k R) \to V\,.$$

If the following conditions hold (cf. Corollary 2.4):

$$\mathrm{Im}(\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) \subseteq R\,, \tag{6}$$
$$\alpha \circ (\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) = -(\beta \otimes \mathrm{Id} - \mathrm{Id} \otimes \beta)\,, \tag{7}$$
$$\beta \circ (\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) = 0\,, \tag{8}$$

then $\operatorname{gr} U \simeq A$ as k-algebras.

Remark 4.2. In stating the theorem, we first defined R and then defined P via the linear maps α and β. Alternatively, one may begin with P and assume that the condition $P \cap \mathcal{F}^1 T(V) = 0$ holds (cf. Corollary 2.2(a)), in which case the linear maps α and β can be defined (cf. Corollary 2.3).

The proof proceeds as follows. First, we construct a (unital) graded deformation $\mathcal{A}_t$ of A over $k[t]$, which is therefore isomorphic to $A[t]$ as a k-vector space. Specifically, the deformed multiplication

$$\mu_{\mathcal{A}_t} = \sum_{i \geq 0} \mu_i t^i\colon A \otimes_k A \to A[t]$$

is constructed to satisfy the following properties:

$$\mu_1|_R = \alpha\,, \qquad \mu_2|_R = \beta\,, \qquad \mu_i|_R = 0 \quad \text{for all} \quad i > 2\,,$$

which allows us to define an algebra morphism from U to the fiber $\mathcal{A}_1$ of $\mathcal{A}_t$ at $t = 1$. This morphism, combined with the flatness of the deformation $\mathcal{A}_t$, will then enable us to show that the morphism $A \to \operatorname{gr} U$ in (1) is injective.

4.1 Construction of the deformation

Let $\pi\colon T(V) \to A$ be the canonical projection, and let $\widetilde{\alpha} \coloneqq \pi \circ \alpha$ and $\widetilde{\beta} \coloneqq \pi \circ \beta$ be maps from R to A. Since $K_2'(A) = R$ and

$$\mathrm{Hom}_{A^e}(K_2(A), A) = \mathrm{Hom}_k(K_2'(A), A)\,,$$

we may regard $\widetilde{\alpha}$ and $\widetilde{\beta}$ as 2-cochains on the Koszul complex. Let $\mu_A\colon A \otimes_k A \to A$ denote the multiplication on A, so that

$$\pi \circ \mu_{T(V)} = \mu_A \circ (\pi \otimes \pi),$$

since π is an algebra morphism. We first point out two simple observations:

- For any $x \in \mathcal{F}^1 T(V)$, one has $\pi(x) = x$. Since the images of α and β lie in $\mathcal{F}^1 T(V)$, we will often identify $\widetilde{\alpha}$ with α (respectively $\widetilde{\beta}$ with β) after restricting the codomain to V (respectively k).
- For any $i, j > 0$, the restriction of the multiplication

  $$\mu_{T(V)}\colon T_i(V) \otimes T_j(V) \to T_{i+j}(V)$$

  is simply the identity map on $V^{\otimes(i+j)}$. The only nontrivial cases occur when $i = 0$ or $j = 0$, in which case $\mu_{T(V)}$ is no longer an identity map, but instead the natural isomorphism $k \otimes_k T_n(V) \simeq T_n(V) \simeq T_n(V) \otimes_k k$.

We now rewrite conditions (a)–(c) in terms of $\widetilde{\alpha}$ and $\widetilde{\beta}$.

• **Condition (a).** Since $\pi(R) = 0$, we obtain:

$$\begin{aligned} 0 &= \pi \circ \mu_{T(V)} \circ (\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) \\ &= \mu_A \circ (\widetilde{\alpha} \otimes \pi - \pi \otimes \widetilde{\alpha}) \\ &= \mu_A \circ (\widetilde{\alpha} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\alpha})\,. \end{aligned} \tag{9}$$

• **Condition (b).** Postcompose both sides of condition (b) with π. The computation on the right-hand side proceeds exactly as in condition (a):

$$\begin{aligned} -\pi \circ \mu_{T(V)} \circ (\beta \otimes \mathrm{Id} - \mathrm{Id} \otimes \beta) &= -\mu_A \circ (\widetilde{\beta} \otimes \pi - \pi \otimes \widetilde{\beta}) \\ &= -\mu_A \circ (\widetilde{\beta} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\beta})\,, \end{aligned}$$

and the left-hand side becomes

$$\begin{aligned} \pi \circ \alpha \circ \mu_{T(V)} \circ (\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) &= \widetilde{\alpha} \circ \mu_{T(V)} \circ (\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) \\ &= \widetilde{\alpha} \circ (\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) \\ &= \widetilde{\alpha} \circ (\widetilde{\alpha} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\alpha})\,. \end{aligned}$$

Combining these results, we obtain:

$$\widetilde{\alpha} \circ (\widetilde{\alpha} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\alpha}) = -\mu_A \circ (\widetilde{\beta} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\beta})\,. \tag{10}$$

• **Condition (c).** An analogous computation to that in condition (b) shows that

$$\begin{aligned} 0 &= \pi \circ \beta \circ \mu_{T(V)} \circ (\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) \\ &= \widetilde{\beta} \circ \mu_{T(V)} \circ (\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) \\ &= \widetilde{\beta} \circ (\alpha \otimes \mathrm{Id} - \mathrm{Id} \otimes \alpha) \\ &= \widetilde{\beta} \circ (\widetilde{\alpha} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\alpha})\,. \end{aligned} \tag{11}$$

Let

$$C^\bullet(A, A) \coloneqq \mathrm{Hom}_{A^e}(B_\bullet(A), A)$$

be the cochain complex on $B_\bullet(A)$, and let

$$\overline{C}^\bullet(A, A) \coloneqq \mathrm{Hom}_{A^e}(\overline{B}_\bullet(A), A) \simeq \left\{ f^\bullet \in C^\bullet(A, A) \mid f^\bullet(\mathrm{Ker}(q_\bullet)) = 0 \right\}$$

be the cochain complex on $\overline{B}_\bullet(A)$, regarded as a subcomplex of $C^\bullet(A, A)$. We also state and prove the

following lemma, which will be needed to establish the unitality of the deformation:

Lemma 4.3. The subcomplex $\overline{C}^\bullet(A,A)$ is closed under the circle product; that is, if $f \in \overline{C}^i(A,A)$ and $g \in \overline{C}^j(A,A)$, then $f \circ g \in \overline{C}^{i+j-1}(A,A)$.

Proof. Omitted. ∎

We now begin the construction of the graded deformation. Following the notation of Lemma 3.2, let $\psi_\bullet \coloneqq \overline{\psi}_\bullet \circ q_\bullet$.

• **First-level deformation.** For any $x = x_1 \otimes x_2 \otimes x_3 \in K_3'(A)$,

$$(d^*\widetilde{\alpha})(x) = \mu_A(x_1 \otimes \widetilde{\alpha}(x_2 \otimes x_3)) - \mu_A(\widetilde{\alpha}(x_1 \otimes x_2) \otimes x_3) = 0$$

by (9). Thus $\widetilde{\alpha}$ is a 2-cocycle on the Koszul complex, and the 2-cocycle

$$\mu_1 \coloneqq \widetilde{\alpha} \circ \psi_2$$

on the bar complex defines the first-level deformation. Since $\psi_2 = \overline{\psi}_2 \circ q_2$, it is clear that $\mu_1 \in \overline{C}^2(A,A)$.

• **Second-level deformation.** An analogous computation to the first-level case yields:

$$d^*\widetilde{\beta} = \mu_A \circ (\mathrm{Id} \otimes \widetilde{\beta} - \widetilde{\beta} \otimes \mathrm{Id}) \overset{(10)}{=} \widetilde{\alpha} \circ (\widetilde{\alpha} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\alpha})\,.$$

Let $\mu_2' \coloneqq \widetilde{\beta} \circ \psi_2$ and

$$\gamma \coloneqq d^*\mu_2' - \mu_1 \circ (\mu_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes \mu_1)\,.$$

We first note that γ is a cocycle since

$$\mu_1 \circ (\mu_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes \mu_1) = \frac{1}{2}[\mu_1, \mu_1]\,,$$

where $[-,-]$ denotes the Gerstenhaber bracket, and $d^*\mu_1 = 0$ implies

$$d^*[\mu_1,\mu_1] = -[d^*\mu_1, \mu_1] + [\mu_1, d^*\mu_1] = 0\,.$$

Since condition (a) implies $\mathrm{Im}(\widetilde{\alpha} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\alpha}) \subseteq R$, precomposing γ with ι_3 yields

$$\begin{aligned}
\gamma\iota_3 &= (d^*\mu_2') \circ \iota_3 - \mu_1 \circ (\mu_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes \mu_1) \circ \iota_3 \\
&= d^*(\mu_2'\iota_2) - \mu_1 \circ (\widetilde{\alpha} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\alpha}) \\
&= d^*\widetilde{\beta} - \widetilde{\alpha} \circ (\widetilde{\alpha} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\alpha}) \\
&= 0\,.
\end{aligned}$$

Thus $\gamma\iota_3$ is a 3-coboundary on the Koszul complex, hence γ is a 3-coboundary on the bar complex. Moreover,

$$(d^*\mu_2')(\mathrm{Ker}(q_3)) \subseteq \mu_2'(\mathrm{Ker}(q_2)) = 0$$

together with Lemma 4.3 implies γ is a coboundary in $\overline{C}^3(A,A)$. Therefore we can choose a 2-cochain $\mu_2'' \in \overline{C}^2(A,A)$ such that $\gamma = d^*\mu_2''$, and set

$$\mu_2 \coloneqq \mu_2' - \mu_2'' + \mu_2''\iota_2\psi_2 \in \overline{C}^2(A,A)\,.$$

Then

$$\mu_2 \circ \iota_2 = \widetilde{\beta}\,, \tag{12}$$

and μ_2 satisfies the Maurer–Cartan equation:

$$\begin{aligned}
d^*\mu_2 &= d^*\mu_2' - d^*\mu_2'' + (d^*\mu_2'')\iota_3\psi_3 \\
&= \mu_1 \circ (\mu_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes \mu_1) + \gamma\iota_3\psi_3 \\
&= \mu_1 \circ (\mu_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes \mu_1)\,.
\end{aligned}$$

• **Third-level deformation.** Consider the following 3-cochain on the bar complex:

$$\delta \coloneqq \mu_1 \circ (\mu_2 \otimes \mathrm{Id} - \mathrm{Id} \otimes \mu_2) + \mu_2 \circ (\mu_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes \mu_1)\,.$$

Again, precomposition by ι_3 yields:

$$\begin{aligned}
\delta\iota_3 &= \mu_1 \circ (\widetilde{\beta} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\beta}) + \mu_2 \circ (\widetilde{\alpha} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\alpha}) \\
&\overset{(12)}{=} \mu_1 \circ (\widetilde{\beta} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\beta}) + \widetilde{\beta} \circ (\widetilde{\alpha} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\alpha}) \\
&\overset{(11)}{=} \mu_1 \circ (\widetilde{\beta} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\beta}) \\
&= 0\,,
\end{aligned}$$

since $\mathrm{Im}(\widetilde{\beta} \otimes \mathrm{Id} - \mathrm{Id} \otimes \widetilde{\beta}) \subseteq k \otimes_k V + V \otimes_k k \subseteq \mathrm{Ker}(q_2)$. Thus $\delta\iota_3$ is a 3-coboundary on the Koszul complex, and equivalently δ is a coboundary in $C^3(A, A)$. Lemma 4.3 again tells that δ is a coboundary in $\overline{C}^3(A, A)$, and we can find $\mu_3 \in \overline{C}^2(A, A)$ satisfying

$$\mu_1 \circ (\mu_2 \otimes \mathrm{Id} - \mathrm{Id} \otimes \mu_2) + \mu_2 \circ (\mu_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes \mu_1) = d^* \mu_3\,.$$

• **Higher-level deformation.** The obstruction to the n-th level graded deformation is given by the graded Hochschild cohomology $HH^{3,-n}(A)$. Since $K_3'(A)$ is homogeneous of degree 3,

$$HH^{3,-n}(A) = 0 \qquad \text{for} \quad n > 3,$$

and therefore the third-level deformation extends to a graded deformation $\mathcal{A}_t$ of A over $k[t]$. Moreover, each obstruction term in the Maurer–Cartan equation lies in $\overline{C}^3(A, A)$ by Lemma 4.3, and thus each 2-cochain can be chosen from $\overline{C}^2(A, A)$. This proves the unitality of the deformation.

4.2 Proof of the Second Part

We first recall some general facts about (unital) graded deformations. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded k-algebra, and let $\mathcal{A}_t$ be a graded deformation of A over $k[t]$. Thus $\mathcal{A}_t = A[t]$ as a $k[t]$-module, and $\mathcal{A}_t$ carries a graded $k[t]$-module structure induced by the grading of A together with the convention $\deg t = 1$:

$$(\mathcal{A}_t)_n = \{\, a_0 + a_1 t + \cdots + a_n t^n \mid \deg a_i = n - i \,\}.$$

We also denote the corresponding filtration on $\mathcal{A}_t$ by

$$\mathcal{F}^n \mathcal{A}_t = \bigoplus_{0 \leq i \leq n} (\mathcal{A}_t)_n = \{\, a_0 + a_1 t + \cdots + a_n t^n \mid \deg a_i \leq n - i \,\}\,.$$

Now consider the fiber

$$\mathcal{A}_c \coloneqq \mathcal{A}_t \otimes_{k[t]} k[t]/(t - c) \simeq \mathcal{A}_t/(t - c)\mathcal{A}_t$$

of $\mathcal{A}_t$ at $c \in k$, which also has a filtration induced from that of $\mathcal{A}_t$. We note that

$$(-) \otimes_{k[t]} k[t]/(t - c) \simeq (-) \otimes_{k[t]} k$$

is a base change functor from $k[t]$ to k via the evaluation map $t \mapsto c$, thus $\mathcal{A}_c$ has a k-algebra structure. Moreover,

$$\mathcal{A}_c \simeq \mathcal{A}_t/(t - c)\mathcal{A}_t \simeq A$$

as a k-vector space. Under this identification, $\mathcal{A}_c$ can be regarded as a k-vector space A endowed with the following multiplication

$$A \otimes_k A \to A\,, \qquad x \otimes y \mapsto \mu_A(x \otimes y) + \sum_{i \geq 1} c^i \mu_i(x \otimes y)\,,$$

and a filtration

$$\mathcal{F}^n\mathcal{A}_c = \mathcal{F}^n A = \bigoplus_{0\le i\le n} A_i\,,$$

where μ_A is the undeformed multiplication of A. Since each μ_i is of degree $-i$, one can see that

$$\operatorname{gr}\mathcal{A}_c \simeq A \tag{13}$$

as graded k-algebras, where A is endowed with the multiplication μ_A.

Now let us return to the proof of the main theorem. Considering V as a subspace of $\mathcal{A}_1$, one can obtain a k-algebra map

$$\phi\colon T(V) \to \mathcal{A}_1$$

which preserves the filtration. We also note that

$$\mathcal{F}^1 T(V) = k \oplus V = \mathcal{F}^1\mathcal{A}_1$$

and restriction of ϕ onto $\mathcal{F}^1T(V)$ becomes the identity map.

We now claim that $P \subseteq \operatorname{Ker}\phi$. To prove this, consider an element $x \in R \subseteq T(V)$. Again, $R \subseteq V \otimes_k V$ can be regarded as a subspace of $T(V)\otimes_k T(V)$, and the restriction of the multiplication map $\mu_{T(V)}$ onto $V\otimes_k V$ becomes the identity map. Therefore

$$\phi(x) = \phi\mu_{T(V)}(x) = \mu_{\mathcal{A}_1}(\phi\otimes\phi)(x) = \mu_{\mathcal{A}_1}(x)\,.$$

Since the multiplication $\mu_{\mathcal{A}_1}$ on the fiber is induced from the multiplication $\mu_{\mathcal{A}_t}$ of the deformation,

$$\begin{aligned}\mu_{\mathcal{A}_t}(x) &= \mu_A(x) + \mu_1(x)t + \mu_2(x)t^2\\ &= 0 + \alpha(x)t + \beta(x)t^2\end{aligned}$$

implies

$$\phi(x) = \mu_{\mathcal{A}_1}(x) = \alpha(x) + \beta(x)\,.$$

Since $\alpha(x) \in V$ and $\beta(x) \in k$, we obtain

$$\phi(x) = \phi(\alpha(x)) + \phi(\beta(x))$$

and therefore $P \subseteq \operatorname{Ker}\phi$.

The above claim implies that ϕ induces a map

$$U = T(V)/(P) \to \mathcal{A}_1\,,$$

and since ϕ preserves filtration, we can further obtain a k-algebra morphism

$$\operatorname{gr} U \to \operatorname{gr}\mathcal{A}_1\,. \tag{14}$$

Finally, consider the following composition of k-algebra morphisms:

$$\theta\colon A \xrightarrow{(1)} \operatorname{gr} U \xrightarrow{(14)} \operatorname{gr}\mathcal{A}_1 \xrightarrow{(13)} A\,.$$

One can check that the restriction of θ onto the subspace $V \subseteq A$ becomes the identity map, and therefore $\theta = \mathrm{Id}$ (due to the universal property of $T(V)$). This completes the proof.

References

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